RELATION AND FUNCTION
CLASS - 12
MATHEMATICS

Relation -

If A and B are two non-empty sets, then a relation R from A to B is a subset of A x B. If R ⊆ A x B and (a, b) ∈ R, then we say that a is related to b by the relation R, written as aRb.

Domain and Range of a Relation-

Let R be a relation from a set A to set B. Then, set of all first components or coordinates of the ordered pairs belonging to R is called : the domain of R, while the set of all second components or coordinates = of the ordered pairs belonging to R is called the range of R.
Thus, domain of R = {a : (a , b) ∈ R} and range of R = {b : (a, b) ∈ R}

Types of Relations-

(i) Void Relation

As Φ ⊂ A x A, for any set A, so Φ is a relation on A, called the empty or void relation.

(ii) Universal Relation

Since, $A \times A \subseteq A \times A$, so $A \times A$ is a relation on A, called the universal relation.

(iii) Identity Relation The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A.

(iv) Reflexive Relation A relation R is said to be reflexive relation, if every element of A is related to itself. Thus, $(a, a) \in R$, $\forall a \in A = R$ is reflexive.

(v) Symmetric Relation

A relation R is said to be symmetric relation, iff $(a, b) \in R$ $(b, a) \in R, \forall a, b \in A$ i.e., $a R b \Rightarrow b R a, \forall a, b \in A \Rightarrow R$ is symmetric.

(vi) Anti-Symmetric Relation A relation R is said to be anti-symmetric relation, iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b, \forall a, b \in A$.

(vii) Transitive Relation A relation R is said to be transitive relation, iff $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$, $\forall a, b, c \in A$

(viii) Equivalence Relation A relation R is said to be an equivalence relation, if it is simultaneously reflexive, symmetric and transitive on A.

(ix) Partial Order Relation A relation R is said to be a partial order relation, if it is simultaneously reflexive, symmetric and anti-symmetric on A.

(x) Total Order Relation A relation R on a set A is said to be a total order relation on A, if R is a partial order relation on A.

Inverse Relation

If A and B are two non-empty sets and R be a relation from A to B, such that R = {(a, b) : a ∈ A, b ∈ B}, then the inverse of R, denoted by R -1 , i a relation from B to A and is defined by R -1 = {(b, a) : (a, b) ∈ R}

Equivalence Classes of an Equivalence Relation

Let R be equivalence relation in A (≠ Φ). Let a ∈ A. Then, the equivalence class of a denoted by [a] or {a} is defined as the set of all those points of A which are related to a under the relation R.

Composition of Relation

Let R and S be two relations from sets A to B and B to C respectively, then we can define relation SoR from A to C such that (a, c) ∈ So R ⇔ ∃ b ∈ B such that (a, b) ∈ R and (b, c) ∈ S. This relation SoR is called the composition of R and S.

(i) RoS ≠ SoR

(ii) (SoR)-1 = R-1 o S-1 known as reversal rule. Congruence Modulo m Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m, if a – b is divisible by m and we write a ≡ b (mod m).

i.e., a ≡ b (mod m) ⇔ a – b is divisible by m.

Important Results on Relation

1. If R and S are two equivalence relations on a set A, then R ∩ S is also on 'equivalence relation on A.
2. The union of two equivalence relations on a set is not necessarily an equivalence relation on the set.
3. If R is an equivalence relation on a set A, then R-1 is also an equivalence relation on A.
4. If a set A has n elements, then number of reflexive relations from A to A is 2n2 – 2
5. Let A and B be two non-empty finite sets consisting of m and n elements, respectively.

Then, A x B consists of mn ordered pairs. So, total number of relations from A to B is 2^nm .

Example 1: Show that the relation R in the set Z of integers given by R = {(a, b) : 2 divides a – b} is an equivalence relation.

Solution. R is reflexive, as 2 divides (a – a) for all a ∈ Z.
Further, if (a, b) ∈ R, then 2 divides a – b.
Therefore, 2 divides b – a.
Hence, (b, a) ∈ R, which shows that R is symmetric.

Similarly, if (a, b) ∈ R and (b, c) ∈ R, then (a – b) and (b – c) are divisible by 2.
Now, a – c = (a – b) + (b – c) is even. (from the above statements)
From this,
(a – c) is divisible by 2.
This shows that R is transitive.
Thus, R is an equivalence relation in Z.

Properties

1. Generally binary operations are represented by the symbols * , +, … etc., instead of letters figure etc.

2.Addition is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively.

3. While addition on the set S of all irrationals is not a binary operation. Multiplication is a binary operation on each one of the sets N, Z, Q, R and C of natural numbers, integers, rationals, real and complex numbers, respectively.

4. While multiplication on the set S of all irrationals is not a binary operation.
5. Subtraction is a binary operation on each one of the sets Z, Q, R and C of integers, rationals, real and complex numbers, respectively.

6. While subtraction on the set of natural numbers is not a binary operation. Let S be a non-empty set and P(S) be its power set. Then, the union and intersection on P(S) is a binary operation.

7. Division is not a binary operation on any of the sets N, Z, Q, R and C. However, it is not a binary operation on the sets of all non-zero rational (real or complex) numbers.

8. Exponential operation (a, b) → a b is a binary operation on set N of natural numbers while it is not a binary operation on set Z of integers.

Types of Binary Operations

(i) Associative Law

A binary operation * on a non-empty set S is said to be associative, if (a * b) * c = a * (b * c), ∀ a, b, c ∈ S. Let R be the set of real numbers, then addition and multiplication on R satisfies the associative law.

(ii) Commutative Law A binary operation * on a non-empty set S is said to be commutative, if a * b = b * a, ∀ a, b ∈ S. Addition and multiplication are commutative binary operations on Z but subtraction not a commutative binary operation, since 2 — 3 ≠ 3— 2 . Union and intersection are commutative binary operations on the power P(S) of all subsets of set S. But difference of sets is not a commutative binary operation on P(S).

(iii) Distributive Law Let * and o be two binary operations on a non-empty sets. We say that * is distributed over o.,
if a * (b o c)= (a * b) o (a * c), ∀ a, b, c ∈ S also called (left distribution) and (b o c) * a = (b * a) o (c * a), ∀ a, b, c ∈ S also called (right distribution).

Let R be the set of all real numbers, then multiplication distributes addition on R.
Since, a.(b + c) = a.b + a.c, ∀ a, b, c ∈ R.

(iv) Identity Element Let * be a binary operation on a non-empty set S. An element e a S, if it exist such that

a * e = e * a = a, ∀ a ∈ S.
is called an identity elements of S, with respect to *. For addition on R, zero is the identity elements in R. Since,

$$a + 0 = 0 + a = a, \ \forall \ a \in R$$

For multiplication on R, 1 is the identity element in R.

Since, a x 1 =1 x a = a, ∀ a ∈ R Let P (S) be the power set of a non-empty set S.
Then, Φ is the identity element for union on P (S) as A ∪ Φ =Φ ∪ A = A, ∀ A ∈ P(S) Also, S is the identity element for intersection on P(S). Since, A ∩ S=A ∩ S=A, ∀ A ∈ P(S).

For addition on N the identity element does not exist. But for multiplication on N the idenitity element is 1.

(v) Inverse of an Element
Let * be a binary operation on a non-empty set 'S' and let 'e' be the identity element.

Let a ∈ S. we say that a-1 is invertible, if there exists an element b ∈ S such that a * b = b * a = e Also, in this case, b is called the inverse of a and we write, a-1 = b Addition on N has no identity element and accordingly N has no invertible element.

Multiplication on N has 1 as the identity element and no element other than 1 is invertible.
Let S be a finite set containing n elements. Then, the total number of binary operations on S in n n2 Let S be a finite set containing n elements.
Then, the total number of commutative binary operation on S is n [n(n+1)/2]

Example 2: Show that subtraction and division are not binary operations on R.

Solution: N × N → N, given by (a, b) → a – b, is not binary operation, as the image of (2, 5) under '–' is 2 – 5 = – 3 ∉ N.
Similarly, ÷: N × N → N, given by (a, b) → a ÷ b is not a binary operation, as the image of (2, 5) under ÷ is 2 ÷ 5 = 2/5 ∉N.

Functions

“A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B”.

A function is a relationship which explains that there should be only one output for each input. It is a special kind of relation(a set of ordered pairs) which obeys a rule, i.e. every y-value should be connected to only one y-value.

Types of Functions

1.One to one Function: A function $f : X \rightarrow Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct, i.e.,
for every $x1$, $x2 \in X$, $f(x1) = f(x2)$ implies $x1 = x2$. Otherwise, f is called many-one.

Onto Function:

A function $f: X \rightarrow Y$ is said to be onto (or surjective), if every element of Y is the image of some element of X under f, i.e., for every $y \in Y$, there exists an element x in X such that $f(x) = y$.

One-one and Onto Function: A function $f: X \rightarrow Y$ is said to be one-one and onto (or bijective), if f is both one-one and onto.

Composition of Functions

Let f: A → B and g: B → C be two functions. Then the composition of f and g, denoted by gof, is defined as the function gof: A → C given by;
gof (x) = g(f (x)), ∀ x ∈ A

Invertible Functions

A function f : X → Y is defined to be invertible if there exists a function g : Y → X such that gof = IX and fog = IY. The function g is called the inverse of f and is denoted by f–1.

An important note is that, if f is invertible, then f must be one-one and onto and conversely, if f is one-one and onto, then f must be invertible.

Example 3: Let f : {2, 3, 4, 5} → {3, 4, 5, 9} and g : {3, 4, 5, 9} → {7, 11, 15} be functions defined as f(2) = 3, f(3) = 4, f(4) = f(5) = 5 and g (3) = g (4) = 7 and g (5) = g (9) = 11. Find gof.

Solution: From the given, we have:
gof(2) = g (f(2)) = g (3) = 7
gof (3) = g (f(3)) = g (4) = 7
gof(4) = g (f(4)) = g (5) = 11
gof(5) = g (5) = 11